

A SURVEY OF DYNAMICAL MATRICES THEORY

LIN ZHANG AND JUNDE WU

ABSTRACT. In this note, we survey some elementary theorems and proofs concerning dynamical matrices theory. Some mathematical concepts and results involved in quantum information theory are reviewed. A little new result on the matrix representation of quantum operation are obtained. And best separable approximation for quantum operations is presented.

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1. INTRODUCTION AND PRELIMINARIES

Positive linear maps on some operator algebras are a very important subject of both the mathematical and the physical literature for several years. The images of positive operators acting on a given Hilbert space under such a map are positive operators acting on the same Hilbert space. A map Φ is called k -positive for some $k \in \mathbb{N}$ if the tensor product $\Phi \otimes \text{Id}_k$ is positive. We call Φ is a *completely positive* (CP) when it is k -positive for any $k \in \mathbb{N}$. Completely positive maps (CP maps, for short) describe the dynamics of open quantum systems. The structure of the set of CP maps is well understood due to the theorems of Stinespring [12], Kraus [8], and Choi [3]. Choi's theorem is also proved by another simple approach in [11].

In this paper, only finite dimensional complex vector spaces are considered. A column vector in a complex vector space is denoted by $|\phi\rangle$, the symbol ϕ is a label, while $|\cdot\rangle$ denotes that the object is a complex column vector. This notation for complex vectors is called *Dirac notation*. Throughout the paper, \dagger , t and $*$ stand for Hermitian conjugate, transposition and complex conjugate, respectively, of matrices with respect to a given orthonormal basis. Given a vector

$$|\phi\rangle = [\phi_1, \phi_2, \dots, \phi_d]^t,$$

its dual is defined as

$$\langle\phi| = [\phi_1^* \ \phi_2^* \ \cdots \ \phi_d^*] \equiv (|\phi\rangle)^\dagger.$$

Given the vectors $|\phi\rangle, |\varphi\rangle$, the inner product between two vectors is denoted by $\langle\phi|\varphi\rangle$, which is defined as follows:

$$\langle\phi|\varphi\rangle \equiv \sum_{i=1}^d \phi_i^* \varphi_i = [\phi_1^* \ \phi_2^* \ \cdots \ \phi_d^*][\phi_1, \phi_2, \dots, \phi_d]^t.$$

The *norm* of a vector $|\phi\rangle$ is defined as $\|\phi\| = \sqrt{\langle\phi|\phi\rangle}$. Unite vectors are those vectors with unit norm. Two vectors are *orthogonal* if they have zero product. The *outer product* of the given vectors $|\phi\rangle$ and $|\varphi\rangle$ is given by

$$|\phi\rangle\langle\varphi| \equiv \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_d \end{bmatrix} [\varphi_1^* \ \varphi_2^* \ \cdots \ \varphi_d^*] = \begin{bmatrix} \phi_1\varphi_1^* & \phi_1\varphi_2^* & \cdots & \phi_1\varphi_d^* \\ \phi_2\varphi_1^* & \phi_2\varphi_2^* & \cdots & \phi_2\varphi_d^* \\ \vdots & \vdots & \vdots & \vdots \\ \phi_d\varphi_1^* & \phi_d\varphi_2^* & \cdots & \phi_d\varphi_d^* \end{bmatrix}.$$

A set of vectors $\{|v_k\rangle\}_{k=1}^n$ in a vector space \mathcal{V} is *orthonormal* if the vectors are normalized and orthogonal, that is, $\langle v_i|v_j\rangle = \delta_{ij}$. If, in addition, $n = \dim \mathcal{V}$, this set of vectors form an orthonormal basis for \mathcal{V} . Here we have a simple but useful fact that $\sum_{k=1}^n |v_k\rangle\langle v_k| = I_n$ for given an orthonormal basis $\{|v_k\rangle\}_{k=1}^n$ in a vector space \mathcal{V} . This called the *completeness relation*.

Quantum states will now be introduced. A *quantum system* is a physical system that obeys the laws of quantum mechanics. Let us assume that we are given two quantum systems. The first one is owned by Alice, and the second one by Bob. The physical *states* of Alice's system may be described by states in a Hilbert space \mathcal{H}_A of dimension $d_A = N$, and in Bob's system in a Hilbert space \mathcal{H}_B of dimension $d_B = M$. The *tensor product* is a ubiquitous mathematical operation which can be used

to combine vector spaces to form a larger vector space. Given two vector spaces \mathcal{V} and \mathcal{W} , we can combine them to form the vector space $\mathcal{V} \otimes \mathcal{W}$, with $\dim(\mathcal{V} \otimes \mathcal{W}) = \dim(\mathcal{V}) \times \dim(\mathcal{W})$. The bipartite quantum system is then described by vectors in the tensor-product of the two spaces $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and $\dim(\mathcal{H}) = d_A d_B$. A *pure state* of dimension d can be represented by a d -dimensional complex unit vector $|\psi\rangle$. For real θ , the vectors $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same state. More generally, a d -dimensional quantum state is represented by a $d \times d$ complex matrix ρ , also called a *density matrix*, which is a non-negative linear operator, acting on a complex Hilbert space \mathcal{H} , with trace 1. A pure state can be represented either by its state vector $|\psi\rangle$, or by its density matrix $\rho = |\psi\rangle\langle\psi|$. States which are not pure are called *mixed states*. A simple test for whether a state ρ is pure or mixed is to take the trace of ρ^2 : $\text{tr}(\rho^2) = 1$ if ρ is pure and $\text{tr}(\rho^2) < 1$ if ρ is mixed. A mixed state can be expressed as a mixture of pure states in many different ways.

Suppose that $|v\rangle \in \mathcal{V}$, $|w\rangle \in \mathcal{W}$. The vector $|v\rangle \otimes |w\rangle \in \mathcal{V} \otimes \mathcal{W}$. The vector $|v\rangle \otimes |w\rangle$ is computed as follows:

$$|v\rangle \otimes |w\rangle = \begin{bmatrix} w_1|v\rangle \\ \vdots \\ w_k|v\rangle \\ \vdots \\ w_n|v\rangle \end{bmatrix} \quad \text{if } |w\rangle = \begin{bmatrix} w_1 \\ \vdots \\ w_k \\ \vdots \\ w_{d_B} \end{bmatrix} \quad \text{and } |v\rangle = \begin{bmatrix} v_1 \\ \vdots \\ v_k \\ \vdots \\ v_{d_A} \end{bmatrix}.$$

Similarly, the tensor product of two given matrices will be explained as follows: with the orthonormal bases $\{|m\rangle\}(m = 1, \dots, d_A)$ of \mathcal{H}_A and $\{|\mu\rangle\}(\mu = 1, \dots, d_B)$ of \mathcal{H}_B , respectively, the orthonormal basis of \mathcal{H} can be described as $\{|m\rangle \otimes |\mu\rangle \equiv |m\mu\rangle\}(m = 1, \dots, d_A; \mu = 1, \dots, d_B)$ (throughout the present paper, Roman indices correspond to the subsystem A and Greek indices to the subsystem B .) for which two types of ordering are very important such as:

(i) Ordering of type-I:

$$\{|11\rangle, |21\rangle, \dots, |d_A 1\rangle; \dots; |1\mu\rangle, |2\mu\rangle, \dots, |d_A \mu\rangle; \dots; |1d_B\rangle, |2d_B\rangle, \dots, |d_A d_B\rangle\}.$$

(ii) Ordering of type-II:

$$\{|11\rangle, |12\rangle, \dots, |1d_B\rangle; \dots; |m1\rangle, |m2\rangle, \dots, |md_B\rangle; \dots; |d_A 1\rangle, |d_A 2\rangle, \dots, |d_A d_B\rangle\}.$$

$\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$ means that the set of all bounded linear operators on \mathcal{H} , \mathcal{H}_A and \mathcal{H}_B , respectively. If $X \in \mathcal{B}(\mathcal{H}_A)$ and $Y \in \mathcal{B}(\mathcal{H}_B)$, then $X \otimes Y \in \mathcal{B}(\mathcal{H})$. Suppose that the matrix-representations $X \equiv [x_{mn}]$ and $Y \equiv [y_{\mu\nu}]$ for X and Y with respect to the given orthonormal bases $\{|m\rangle\}_{m=1}^{d_A}$ and $\{|\mu\rangle\}_{\mu=1}^{d_B}$ are given, respectively. Then there are several different matrix-representations of $X \otimes Y$ with respect to the corresponding orthonormal bases of different orderings. For the ordering of type-I, the matrix representation of $X \otimes Y$ is

$$X \otimes Y \equiv \begin{bmatrix} y_{11}X & y_{12}X & \cdots & y_{1d_B}X \\ y_{21}X & y_{22}X & \cdots & y_{2d_B}X \\ \vdots & \vdots & \vdots & \vdots \\ y_{d_B 1}X & y_{d_B 2}X & \cdots & y_{d_B d_B}X \end{bmatrix};$$

while for the ordering of type-II, the matrix representation of $X \otimes Y$ is

$$X \otimes Y \equiv \begin{bmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1d_A}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2d_A}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{d_A1}Y & x_{d_A2}Y & \cdots & x_{d_Ad_A}Y \end{bmatrix}.$$

The ordering of type-I will be employed throughout the present paper if unspecified. For tensor product, we have the following rules: given two matrices S and T acting on vector spaces \mathcal{V} and \mathcal{W} , respectively, vectors $|x\rangle \in \mathcal{V}$ and $|y\rangle \in \mathcal{W}$, then

$$(S \otimes T)(|v\rangle \otimes |w\rangle) = (S|x\rangle) \otimes (T|y\rangle), \text{Tr}(S \otimes T) = \text{tr}(S)\text{tr}(T), (S \otimes T)^\dagger = S^\dagger \otimes T^\dagger.$$

If X, Y act also on \mathcal{V}, \mathcal{W} respectively, we have $(S \otimes T)(X \otimes Y) = SX \otimes TY$. Obviously, tensor product is a bi-linear map.

The description of *subsystems* of a composite quantum system is provided by the *reduced density operator*, which is so useful as to be virtually indispensable in the analysis of composite quantum systems. Suppose we have physical systems A and B , whose state is described by a density operator ρ_{AB} . The state space of the composite quantum system AB is denoted by $\mathcal{D}(\mathcal{H})$, similarly, $\mathcal{D}(\mathcal{H}_A)$ for subsystem A and $\mathcal{D}(\mathcal{H}_B)$ for subsystem B . The reduced density operator for system A is defined by $\text{Tr}_B(\rho_{AB}) \equiv \rho_A$, where Tr_B is a map of operators know as the *partial trace* over system B . The partial trace is defined by

$$\text{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \text{tr}(|b_1\rangle\langle b_2|),$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A , and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B . The trace operation appearing on the right hand side is the usual trace operation for system B , so $\text{tr}(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$. In fact, $\text{Tr}_A = \text{tr} \otimes \text{Id}_B$, $\text{Tr}_B = \text{Id}_A \otimes \text{tr}$ and $\text{Tr} = \text{tr} \otimes \text{tr}$. We have defined the partial trace operation only on a special subclass of operators on AB . More generally, for any matrix Z acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, we have a block construction on Z : $Z = [Z_{\mu\nu}]$ ($\mu, \nu = 1, \dots, d_B = \dim \mathcal{H}_B$), where each $Z_{\mu\nu}$ is a scalar matrix of size $d_A \times d_A$ ($d_A = \dim \mathcal{H}_A$). Therefore

$$Z = \sum_{\mu, \nu=1}^{d_B} Z_{\mu\nu} \otimes |\mu\rangle\langle \nu|.$$

Now the partial trace over system A is provided by

$$\text{Tr}_A(Z) = \sum_{\mu, \nu=1}^{d_B} \text{tr}(Z_{\mu\nu}) |\mu\rangle\langle \nu| \equiv [\text{tr}(Z_{\mu\nu})],$$

while the partial trace over system B is given by

$$\text{Tr}_B(Z) = \sum_{\mu, \nu=1}^{d_B} Z_{\mu\nu} \text{tr}(|\mu\rangle\langle \nu|) = \sum_{\mu=1}^{d_B} Z_{\mu\mu}.$$

The partial trace over the composite quantum system AB is

$$\text{Tr}(Z) = \sum_{\mu=1}^{d_B} \text{tr}(Z_{\mu\mu}).$$

The quantum operations formalism is a general tool for describing the evolution of quantum systems in a wide variety of circumstances, including stochastic changes to quantum states. A simple example of a state change in quantum mechanics is the unitary evolution experienced by a closed quantum system. The final state of the system is related to the initial state by a unitary transformation U ,

$$\rho \rightarrow \mathcal{E}(\rho) = U\rho U^\dagger.$$

Unitary evolution is not the most general type of state change possible in quantum mechanics. Other state changes, described without unitary transformations, arise when a quantum system is coupled to an environment or when a measurement is performed on the system. This formalism is described in detail by Kraus. In this formalism there is an input state and an output state, which are connected by a map

$$\rho \rightarrow \frac{\mathcal{E}(\rho)}{\text{tr}[\mathcal{E}(\rho)]}.$$

This map is determined by a *quantum operation* \mathcal{E} , a linear, trace-decreasing map that preserves positivity. The trace in the denominator is included in order to preserve the trace condition $\text{tr}(\rho) = 1$. The most general form for \mathcal{E} that is physically reasonable, can be shown to be

$$\mathcal{E}(\rho) = \sum_j \Gamma_j \rho \Gamma_j^\dagger.$$

The system operators Γ_j , which must satisfy $\sum_j \Gamma_j \Gamma_j^\dagger \leq I$, completely specify the quantum operation. Formally, every quantum operation has to be described mathematically by a completely positive complex-linear mapping \mathcal{E} , which satisfies $\text{tr}(\mathcal{E}(\rho)) \leq 1$ for all state ρ . A quantum operation is called *quantum channel* if it is trace-preserving.

Given quantum operation $\mathcal{E}, \mathcal{E}_A$, and \mathcal{E}_B on corresponding bipartite quantum system with subsystems A and B , subsystems A , and B , respectively, owing to Jamiołkowski isomorphism, the notion of entanglement can be extended from quantum states to quantum operations. A quantum operation acting on two subsystems is said to be *separable* if its action can be expressed in the Kraus form

$$\mathcal{E}(\cdot) = \sum_k (\Lambda_k^A \otimes \Lambda_k^B) \cdot (\Lambda_k^A \otimes \Lambda_k^B)^\dagger,$$

where Λ_k^A and Λ_k^B are operators acting on each subsystem and they satisfy that

$$\sum_k (\Lambda_k^A \otimes \Lambda_k^B)^\dagger (\Lambda_k^A \otimes \Lambda_k^B) \leq I_A \otimes I_B.$$

Otherwise, it is entangled. When the equality is valid, there is a concept of *separable quantum channel*.

2. VECTORIZATION AND REALIGNMENT OF MATRICES

Definition 2.1. Representation of matrices as vectors on a higher dimensional Hilbert space is called *vectorization*. It transforms a $p \times q$ matrix G into $pq \times 1$ column vector denoted by $|G\rangle\rangle$, this is done by ordering matrix elements, i.e., by stacking the columns of G to form a vector: for example, with a $p \times q$ matrix $G = [g_{ij}]$, $|G\rangle\rangle$ is described as

$$|G\rangle\rangle = \begin{bmatrix} G(\cdot, 1) \\ \vdots \\ G(\cdot, q) \end{bmatrix}, \text{ where } G(\cdot, j) = \begin{bmatrix} g_{1j} \\ \vdots \\ g_{pj} \end{bmatrix} \quad (j = 1, \dots, q).$$

That is, $G(\cdot, j)$ is the j th column vector of matrix G . Dually, $\langle\langle G|$ is a $1 \times pq$ row vector defined as $(|G\rangle\rangle)^\dagger$, i.e., $\langle\langle G| = (|G\rangle\rangle)^\dagger$. (see [4])

Remark 2.2. (i) Vectorization is obviously linear: for matrices S_k and scalars λ_k ,

$$|\sum_k \lambda_k S_k\rangle\rangle = \sum_k \lambda_k |S_k\rangle\rangle.$$

- (ii) Vectorization is inner-product-preserving; i.e. isometry. The Hilbert-Schmidt inner product is equivalent to the usual Euclidean inner product of vectors: for square matrices S, T of the same size, $\langle S, T \rangle = \text{tr}(S^\dagger T) = \langle\langle S | T \rangle\rangle$. It is easily shown that vectorization is one-one and onto. Therefore vectorization is a unitary transformation from Hilbert-Schmidt matrix space to Hilbert vector space.
- (iii) Vectorization is intrinsically related to the tensor product. Consider a square matrix of size $p \times p$, representing an operator acting on the p -dimensional Hilbert space \mathcal{K} . Let $\{|j\rangle\}_{j=1}^p$ be the orthonormal basis of \mathcal{K} for which $|j\rangle$ is column vector with all entries 0 except for j th entry 1. A matrix $T = [t_{ij}] = \sum_{i,j=1}^p t_{ij} E_{ij}$, where $E_{ij} = |i\rangle\langle j|$, is transformed to the vector

$$\begin{aligned}
 |T\rangle\rangle &= |\sum_{i,j=1}^p t_{ij} E_{ij}\rangle\rangle = \sum_{i,j=1}^p t_{ij} |E_{ij}\rangle\rangle = \sum_{i,j=1}^p t_{ij} |i\rangle|j\rangle = \sum_{j=1}^p (\sum_{i=1}^p t_{ij} |i\rangle) |j\rangle \\
 &= \sum_{j=1}^p (T|j\rangle) |j\rangle = (T \otimes I_p) (\sum_{j=1}^p |j\rangle|j\rangle) = (T \otimes I_p) (\sum_{j=1}^p |E_{jj}\rangle\rangle) \\
 (2.1) \quad &= (T \otimes I_p) |\sum_{j=1}^p E_{jj}\rangle\rangle = (T \otimes I_p) |I_p\rangle\rangle = (I_p \otimes T^\dagger) |I_p\rangle\rangle.
 \end{aligned}$$

Thus it follows from the identity above that, for any matrices Q, X and R of the same size $p \times p$,

$$\begin{aligned}
 |QXR\rangle\rangle &= (QXR) \otimes I_p |I_p\rangle\rangle = (Q \otimes I_p)(X \otimes I_p)[(R \otimes I_p)|I_p\rangle\rangle] \\
 &= (Q \otimes I_p)(X \otimes I_p)[(I_p \otimes R^\dagger)|I_p\rangle\rangle] = (Q \otimes I_p)[(X \otimes I_p)(I_p \otimes R^\dagger)]|I_p\rangle\rangle \\
 &= (Q \otimes I_p)[(I_p \otimes R^\dagger)(X \otimes I_p)]|I_p\rangle\rangle = (Q \otimes I_p)(I_p \otimes R^\dagger)|X\rangle\rangle \\
 (2.2) \quad &= Q \otimes R^\dagger |X\rangle\rangle
 \end{aligned}$$

and

$$(2.3) \quad |XY\rangle\rangle = (X \otimes I_p)|Y\rangle\rangle = (I_p \otimes Y^\dagger)|X\rangle\rangle.$$

- (iv) For any matrix Y ,

$$(2.4) \quad \langle\langle Y^* | = (|Y^*\rangle\rangle)^\dagger = (|Y\rangle\rangle)^{* \dagger} = (|Y\rangle\rangle)^\dagger.$$

- (v) For $S \in \mathcal{B}(\mathcal{H}_A)$ and $T \in \mathcal{B}(\mathcal{H}_B)$, where $\mathcal{H}_A = \mathcal{H}_B$ are d -dimensional Hilbert spaces. For the matrix representations $S = [s_{ij}]$ and $T = [t_{ij}]$ ($i, j = 1, \dots, d$), we have $\text{tr}_B(|S\rangle\rangle\langle\langle T|) = ST^\dagger$ and

$\text{tr}_A(|S\rangle\langle\langle T|) = S^\dagger T^*$. Indeed,

$$\begin{aligned}
\text{tr}_B(|S\rangle\langle\langle T|) &= \sum_{m,n,\mu,\nu=1}^d s_{mn} t_{\mu\nu}^* \text{tr}_B(|mn\rangle\langle\mu\nu|) = \sum_{m,n,\mu,\nu=1}^d s_{mn} t_{\mu\nu}^* \text{tr}_B(|m\rangle\langle\mu| \otimes |n\rangle\langle\nu|) \\
&= \sum_{m,n,\mu,\nu=1}^d s_{mn} t_{\mu\nu}^* \delta_{n\nu} |m\rangle\langle\mu| = \sum_{m,n,\mu=1}^d s_{mn} t_{\mu n}^* |m\rangle\langle\mu| \\
&= \sum_{n=1}^d \left(\sum_{m=1}^d s_{mn} |m\rangle \right) \left(\sum_{\mu=1}^d t_{\mu n} |\mu\rangle \right)^\dagger = \sum_{n=1}^d (S|n\rangle)(T|n\rangle)^\dagger \\
&= \sum_{n=1}^d S|n\rangle\langle n| T^\dagger = S T^\dagger.
\end{aligned}$$

The other identity goes similarly.

Definition 2.3. Let Z be an $d_B \times d_B$ block matrix with each entry of size $d_A \times d_A$; i.e. $Z = [Z_{\mu\nu}]$ represent an operator acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. We define a realigned matrix $\mathcal{R}(Z)$, acting from $\mathcal{H}_B \otimes \mathcal{H}_B$ to $\mathcal{H}_A \otimes \mathcal{H}_A$, of size $d_A^2 \times d_B^2$ that contains the same elements as Z but in different position as

$$\mathcal{R}(Z) = [|Z_{11}\rangle\rangle, \dots, |Z_{d_B 1}\rangle\rangle; \dots; |Z_{1d_B}\rangle\rangle, \dots, |Z_{d_B d_B}\rangle\rangle].$$

In fact, $\mathcal{R}(Z)_{\mu\nu}^{mn} = Z_{\mu\nu}^{mn}$. Similarly, we can also define another alignment \mathcal{R}' as $\mathcal{R}'(Z)_{\mu\nu}^{mn} = Z_{\mu\nu}^{nm}$. Note that alignment of matrices is a one-one linear mapping from the matrix space $\mathcal{M}_{d_A d_B \times d_A d_B}(\mathbb{C})$ onto the matrix space $\mathcal{M}_{d_A^2 \times d_B^2}(\mathbb{C})$.

Proposition 2.4. For a tensor matrix $X \otimes Y$ with the factor matrix X of size $d_A \times d_A$ and the factor matrix $Y = [y_{\mu\nu}]$ of size $d_B \times d_B$, $Z = [y_{\mu\nu} X] = [Z_{\mu\nu}]$. We have:

$$(2.5) \quad \mathcal{R}(X \otimes Y) = |X\rangle\rangle\langle\langle Y^*|.$$

Moreover, a nonzero matrix Z can be factorized as $X \otimes Y$ if and only if $\text{rank}[\mathcal{R}(Z)] = 1$.

Proof.

$$\begin{aligned}
\mathcal{R}(X \otimes Y) &= [|y_{11}X\rangle\rangle, \dots, |y_{d_B 1}X\rangle\rangle; \dots; |y_{1d_B}X\rangle\rangle, \dots, |y_{d_B d_B}X\rangle\rangle] \\
&= [y_{11}|X\rangle\rangle, \dots, y_{d_B 1}|X\rangle\rangle; \dots; y_{1d_B}|X\rangle\rangle, \dots, y_{d_B d_B}|X\rangle\rangle] \\
&= |X\rangle\rangle[y_{11}, \dots, y_{d_B 1}; \dots; y_{1d_B}, \dots, y_{d_B d_B}] = |X\rangle\rangle(|Y\rangle\rangle)^\dagger \\
&= |X\rangle\rangle\langle\langle Y^*|.
\end{aligned}$$

□

For a general block matrix Z , it holds that

$$\begin{aligned}
\mathcal{R}(Z) &= \mathcal{R}\left(\sum_{\mu,\nu=1}^{d_B} Z_{\mu\nu} \otimes |\mu\rangle\langle\nu|\right) = \sum_{\mu,\nu=1}^{d_B} \mathcal{R}(Z_{\mu\nu} \otimes |\mu\rangle\langle\nu|) \\
(2.6) \quad &= \sum_{\mu,\nu=1}^{d_B} |Z_{\mu\nu}\rangle\rangle(|\mu\nu\rangle)^\dagger = \sum_{\mu,\nu=1}^{d_B} |Z_{\mu\nu}\rangle\rangle\langle\langle\mu\nu|.
\end{aligned}$$

Before the properties of realignment derived, we need to know one useful operator called *swap operator*, defined as $S = \sum_{i,j=1}^{N^2} |ij\rangle\langle ji|$, acting on $\mathcal{H}_N \otimes \mathcal{H}_N$. Then by simple computations, we have:

Proposition 2.5. *For any X and Y of the same size $N \times N$. We have:*

(i) S is self-adjoint, unitary, symmetric, and orthogonal;

(ii) $|X^t\rangle\rangle = S|X\rangle\rangle$, $L_{\mathcal{T}} = S$;

(iii) $S(X \otimes Y)S = Y \otimes X$.

Definition 2.6. With S as above, the *flip* transformation of matrices over a bipartite quantum system is defined as

$$\mathcal{F}(Z) = SZS \text{ with } \mathcal{F}(Z)_{nv}^{mu} = Z_{vn}^{um}.$$

Similarly, we can define two *partial flips* as $\mathcal{F}_r(Z) = SZ$ with $\mathcal{F}_r(Z)_{nv}^{mu} = Z_{nv}^{um}$ and $\mathcal{F}_c(Z) = ZS$ with $\mathcal{F}_c(Z)_{nv}^{mu} = Z_{vn}^{um}$ (where ‘r’ and ‘c’ mean that row and column, respectively). Later, we will see that $L_{\mathcal{F}} = S \otimes S$.

Lemma 2.7. ([6]) *Given any two square matrices X, Y of the same size, we have the following equation:*

$$(2.7) \quad |X \otimes Y\rangle\rangle = (I \otimes S \otimes I)|X\rangle\rangle|Y\rangle\rangle.$$

Proposition 2.8. (i) *If X, Y are matrices of the same size $N \times N$, then*

$$(2.8) \quad |\mathcal{R}(X \otimes Y)\rangle\rangle = |X\rangle\rangle|Y\rangle\rangle;$$

i.e., the vectorization of the matrix $|X\rangle\rangle\langle\langle Y^|$ is $|X\rangle\rangle|Y\rangle\rangle$.*

(ii) *Let Z be a matrix of size $N^2 \times N^2$. Then: $|\mathcal{R}(Z)\rangle\rangle = I \otimes S \otimes I|Z\rangle\rangle$, thus $L_{\mathcal{R}} = I \otimes S \otimes I$.*

(iii) *If $\Omega(\cdot) = \sum_{i,j=1}^N (I \otimes |i\rangle\langle j|) \cdot (|i\rangle\langle j| \otimes I)$, then: for any matrices X, Y of the same size $N \times N$,*

$$(2.9) \quad \Omega(|X\rangle\rangle\langle\langle Y|) = X \otimes Y^* \text{ and } \Omega(X \otimes Y^*) = |X\rangle\rangle\langle\langle Y| = \mathcal{R}(X \otimes Y^*).$$

More generally, we have $\Omega(Z) = \mathcal{R}(Z)$ for any matrix Z of size $N^2 \times N^2$.

Proof. (i) and (ii) follow easily from Lemma 2.7.

(iii) Together with Lemma 2.7, it follows from (i) that

$$\begin{aligned} & |\Omega(X \otimes Y^*)\rangle\rangle \\ &= \left| \sum_{i,j=1}^N (I \otimes |i\rangle\langle j|)X \otimes Y^*(|i\rangle\langle j| \otimes I) \right\rangle\rangle = \sum_{i,j=1}^N |(I \otimes |i\rangle\langle j|)X \otimes Y^*(|i\rangle\langle j| \otimes I)| \\ &= \sum_{i,j=1}^N (I \otimes |i\rangle\langle j|) \otimes (|j\rangle\langle i| \otimes I) |X \otimes Y^*\rangle\rangle = \sum_{i,j=1}^N (I \otimes |ij\rangle\langle ji| \otimes I) |X \otimes Y^*\rangle\rangle \\ &= (I \otimes S \otimes I) |X \otimes Y^*\rangle\rangle = |X\rangle\rangle|Y^*\rangle\rangle = |\mathcal{R}(X \otimes Y^*)\rangle\rangle. \end{aligned}$$

Hence $\Omega(X \otimes Y^*) = \mathcal{R}(X \otimes Y^*) = |X\rangle\rangle\langle\langle Y|$. By simple computations, we have also $\Omega(|X\rangle\rangle\langle\langle Y|) = X \otimes Y^*$. Since $\langle m\mu | \mathcal{R}(Z) | n\nu \rangle = \mathcal{R}(Z)_{n\nu}^{m\mu} = Z_{\mu\nu}^{mn}$ and

$$\langle m\mu | \sum_{i,j=1}^N (I \otimes |i\rangle\langle j|) Z (|i\rangle\langle j| \otimes I) | n\nu \rangle = \sum_{i,j=1}^N \delta_{\mu i} \delta_{n j} \langle m j | Z | i \nu \rangle = \langle mn | Z | \mu \nu \rangle = Z_{\mu\nu}^{mn},$$

i.e., $\Omega(Z) = \mathcal{R}(Z)$. In such a way, we obtain the explicit expression for the realignment transformation:

$$(2.10) \quad \mathcal{R}(Z) = \sum_{i,j=1}^N (I \otimes |i\rangle\langle j|) Z (|i\rangle\langle j| \otimes I)$$

for any matrix Z of size $N^2 \times N^2$. \square

Next the relationship among *the realignment*, *the transposition*, and *the flip* over a bipartite quantum system will be discussed. First recall that the transposition \mathcal{T} over bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$ are defined as $\mathcal{T}(Z) \equiv \mathcal{T}_A \otimes \mathcal{T}_B(Z)$ with $\mathcal{T}(Z)_{n\nu}^{m\mu} = Z_{\mu\mu}^{n\nu}$, where \mathcal{T}_A and \mathcal{T}_B are the transpositions with respect to subsystems A and B, respectively. Apparently, $\mathcal{T}_A(Z)_{n\nu}^{m\mu} = Z_{\mu\nu}^{n\mu}$ and $\mathcal{T}_B(Z)_{n\nu}^{m\mu} = Z_{\mu\mu}^{m\nu}$.

Proposition 2.9. (i) \mathcal{T}, \mathcal{R} and \mathcal{F} all are involution; i.e., $\mathcal{T} \circ \mathcal{T} = \mathcal{R} \circ \mathcal{R} = \mathcal{F} \circ \mathcal{F} = Id$.

(ii) $\mathcal{F} \circ \mathcal{T} = \mathcal{T} \circ \mathcal{F}$, $\mathcal{T} \circ \mathcal{R} \neq \mathcal{R} \circ \mathcal{T}$ and $\mathcal{F} \circ \mathcal{R} \neq \mathcal{R} \circ \mathcal{F}$, where \circ stands for the composite of transformations.

(iii) $\mathcal{T} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{F}$ and $\mathcal{R} \circ \mathcal{T} = \mathcal{F} \circ \mathcal{R}$.

(iv) $\mathcal{R}' = \mathcal{T} \circ \mathcal{R} \circ \mathcal{T} = \mathcal{F} \circ \mathcal{R} \circ \mathcal{F}$.

(v) $\mathcal{F}_r = \mathcal{R} \circ \mathcal{T}_A \circ \mathcal{R}$ and $\mathcal{F}_c = \mathcal{R} \circ \mathcal{T}_B \circ \mathcal{R}$.

Proof. It is trivially by some computations. For example, $[\mathcal{T} \circ \mathcal{R}(X)]_{n\nu}^{m\mu} = [\mathcal{T}(X)]_{\mu\nu}^{mn} = X_{mn}^{\mu\nu}$ and $[\mathcal{R} \circ \mathcal{F}(X)]_{n\nu}^{m\mu} = [\mathcal{R}(X)]_{\nu n}^{\mu m} = X_{mn}^{\mu\nu}$; i.e., $[\mathcal{T} \circ \mathcal{R}(X)]_{n\nu}^{m\mu} = [\mathcal{R} \circ \mathcal{F}(X)]_{n\nu}^{m\mu}$ which means that $\mathcal{T} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{F}$. Others go similarly. \square

3. DYNAMICAL MATRICES FOR QUANTUM OPERATIONS

A density matrix

$$(3.1) \quad \rho = \begin{bmatrix} \rho_{11} & \cdots & \rho_{1d_B} \\ \vdots & \vdots & \vdots \\ \rho_{d_B 1} & \cdots & \rho_{d_B d_B} \end{bmatrix} = [\rho_{\mu\nu}]$$

of size $d_B \times d_B$ may be treated as a vector

$$(3.2) \quad |\rho\rangle\rangle = \begin{bmatrix} \rho(\cdot, 1) \\ \vdots \\ \rho(\cdot, d_B) \end{bmatrix}, \text{ where } \rho(\cdot, \nu) = \begin{bmatrix} \rho_{1\nu} \\ \vdots \\ \rho_{d_B \nu} \end{bmatrix} \quad (\nu = 1, \dots, d_B).$$

Suppose that ρ and σ act on \mathcal{H}_B and \mathcal{H}_A , respectively. The action of a linear super-operator $\Phi : \rho \rightarrow \sigma = \Phi(\rho) = [\sigma_{mn}]$ may thus be represented by a matrix $L_\Phi \equiv L$ of size $d_A^2 \times d_B^2$:

$$(3.3) \quad |\sigma\rangle\rangle = |\Phi(\rho)\rangle\rangle = L|\rho\rangle\rangle \text{ or } \sigma_{mn} = \sum_{\mu, \nu=1}^{d_B} L_{\mu\nu}^{mn} \rho_{\mu\nu}.$$

It can be written concretely as the equation of multiplicity of a *supermatrix* and a *supervector*:

$$(3.4) \quad \begin{bmatrix} \sigma(\cdot, 1) \\ \vdots \\ \sigma(\cdot, n) \\ \vdots \\ \sigma(\cdot, d_A) \end{bmatrix} = \begin{bmatrix} L_{11} & \cdots & L_{1\nu} & \cdots & L_{1d_B} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n1} & \cdots & L_{n\nu} & \cdots & L_{nd_B} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{d_A 1} & \cdots & L_{d_A \nu} & \cdots & L_{d_A d_B} \end{bmatrix} \begin{bmatrix} \rho(\cdot, 1) \\ \vdots \\ \rho(\cdot, \nu) \\ \vdots \\ \rho(\cdot, d_B) \end{bmatrix},$$

where

$$(3.5) \quad L_{n\nu} = [L_{\mu\nu}^{mn}] = \begin{bmatrix} L_{1\nu}^{1n} & \cdots & L_{d_B\nu}^{1n} \\ \vdots & \vdots & \vdots \\ L_{d_A\nu}^{d_A n} & \cdots & L_{d_B\nu}^{d_A n} \end{bmatrix}.$$

One must be caution here that n and ν stand for the block row index and the block column index, respectively; while m and μ stand for row index and column index of each block. Now we give a simple example for a qubit map for later use as follows:

$$(3.6) \quad \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix} = \begin{bmatrix} L_{11}^{11} & L_{21}^{11} & L_{12}^{11} & L_{22}^{11} \\ L_{11}^{21} & L_{21}^{21} & L_{12}^{21} & L_{22}^{21} \\ L_{11}^{12} & L_{21}^{12} & L_{12}^{12} & L_{22}^{12} \\ L_{11}^{22} & L_{21}^{22} & L_{12}^{22} & L_{22}^{22} \end{bmatrix} \begin{bmatrix} \rho_{11} \\ \rho_{21} \\ \rho_{12} \\ \rho_{22} \end{bmatrix}.$$

Theorem 3.1. *The requirement that the image σ is a density matrix, so it is Hermitian, positive with unit trace, impose constraints on the matrix L :*

- (i) $\sigma^\dagger = \sigma \implies L_{\mu\nu}^{mn*} = L_{\nu\mu}^{nm}$.
- (ii) $\sigma \geq 0 \implies [\sum_{\mu,\nu=1}^{d_B} L_{\mu\nu}^{mn} \rho_{\mu\nu}] \geq 0$ for any state $\rho = [\rho_{\mu\nu}]$.
- (iii) $\text{tr}(\sigma) = 1 \implies \sum_{m=1}^{d_A} L_{\mu\nu}^{mm} = \delta_{\mu\nu}$.

Proof. (i) **Step 1:** For state $\rho = |\gamma\rangle\langle\gamma|$ ($\gamma \in \{1, \dots, d_B\}$),

$$(3.7) \quad \rho_{\mu\nu} = \langle\mu|\rho|\nu\rangle = \langle\mu|\gamma\rangle\langle\gamma|\nu\rangle = \delta_{\mu\gamma}\delta_{\nu\gamma}.$$

Then

$$(3.8) \quad \sigma_{mn} = \sum_{\mu,\nu=1}^{d_B} L_{\mu\nu}^{mn} \delta_{\mu\gamma} \delta_{\nu\gamma} = L_{\gamma\gamma}^{mn}.$$

Since $\sigma^\dagger = \sigma$, it implies that $\sigma_{mn} = \sigma_{nm}^*$; i.e., $L_{\gamma\gamma}^{mn} = L_{\gamma\gamma}^{nm*}$ ($\gamma \in \{1, \dots, d_B\}$).

Step 2: Setting

$$(3.9) \quad \rho = \frac{1}{2} [|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|] (\alpha, \beta = 1, \dots, d_B; \alpha \neq \beta),$$

we have

$$(3.10) \quad \rho_{\mu\nu} = \frac{1}{2} [\delta_{\mu\alpha}\delta_{\nu\alpha} + \delta_{\mu\beta}\delta_{\nu\beta} + \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}].$$

Hence

$$(3.11) \quad \sigma_{mn} = \frac{1}{2}[L_{\alpha\alpha}^{mn} + L_{\beta\beta}^{mn} + L_{\alpha\beta}^{mn} + L_{\beta\alpha}^{mn}].$$

From the equation $\sigma_{mn} = \sigma_{nm}^*$, we know

$$L_{\alpha\alpha}^{mn} + L_{\beta\beta}^{mn} + L_{\alpha\beta}^{mn} + L_{\beta\alpha}^{mn} = L_{\alpha\alpha}^{nm*} + L_{\beta\beta}^{nm*} + L_{\alpha\beta}^{nm*} + L_{\beta\alpha}^{nm*};$$

i.e.,

$$(3.12) \quad L_{\alpha\beta}^{mn} + L_{\beta\alpha}^{mn} = L_{\alpha\beta}^{nm*} + L_{\beta\alpha}^{nm*}.$$

Step 3: Letting

$$(3.13) \quad \rho = \frac{1}{2}[|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + \sqrt{-1}|\alpha\rangle\langle\beta| - \sqrt{-1}|\beta\rangle\langle\alpha|],$$

we have

$$(3.14) \quad \rho_{\mu\nu} = \frac{1}{2}[\delta_{\mu\alpha}\delta_{\nu\alpha} + \delta_{\mu\beta}\delta_{\nu\beta} + \sqrt{-1}\delta_{\mu\alpha}\delta_{\nu\beta} - \sqrt{-1}\delta_{\mu\beta}\delta_{\nu\alpha}].$$

Hence

$$(3.15) \quad \sigma_{mn} = \frac{1}{2}[L_{\alpha\alpha}^{mn} + L_{\beta\beta}^{mn} + \sqrt{-1}L_{\alpha\beta}^{mn} - \sqrt{-1}L_{\beta\alpha}^{mn}],$$

which implies that

$$(3.16) \quad \sigma_{nm}^* = \frac{1}{2}[L_{\alpha\alpha}^{nm*} + L_{\beta\beta}^{nm*} - \sqrt{-1}L_{\alpha\beta}^{nm*} + \sqrt{-1}L_{\beta\alpha}^{nm*}].$$

This gives rise to:

$$(3.17) \quad L_{\alpha\beta}^{mn} - L_{\beta\alpha}^{mn} = -L_{\alpha\beta}^{nm*} + L_{\beta\alpha}^{nm*}$$

Combing (3.12) with (3.17) gives that $L_{\alpha\beta}^{mn} = L_{\beta\alpha}^{nm*}$.

(ii) is trivial.

(iii) Because $\text{tr}(\sigma) = 1$, that is,

$$(3.18) \quad 1 = \sum_{m=1}^{d_A} \sigma_{mm} = \sum_{m=1}^{d_A} \sum_{\mu, \nu=1}^{d_B} L_{\mu\nu}^{mm} \rho_{\mu\nu}.$$

Step 1: Given $\rho = |\gamma\rangle\langle\gamma|$ ($\gamma \in \{1, \dots, d_B\}$). So $\rho_{\mu\nu} = \langle\mu|\rho|\nu\rangle = \delta_{\mu\gamma}\delta_{\nu\gamma}$. From the equation (3.18), we have that

$$(3.19) \quad 1 = \sum_{m=1}^{d_A} \sum_{\mu, \nu=1}^{d_B} L_{\mu\nu}^{mm} \delta_{\mu\gamma} \delta_{\nu\gamma} = \sum_{m=1}^{d_A} L_{\gamma\gamma}^{mm} (\gamma \in \{1, \dots, d_B\}).$$

Step 2: From the equation (3.9), (3.10) and (3.18), we have that

$$1 = \frac{1}{2} \left[\sum_{m=1}^{d_A} L_{\alpha\alpha}^{mm} + \sum_{m=1}^{d_A} L_{\beta\beta}^{mm} + \sum_{m=1}^{d_A} L_{\alpha\beta}^{mm} + \sum_{m=1}^{d_A} L_{\beta\alpha}^{mm} \right];$$

i.e.,

$$(3.20) \quad \sum_{m=1}^{d_A} L_{\alpha\beta}^{mm} + \sum_{m=1}^{d_A} L_{\beta\alpha}^{mm} = 0.$$

Step 3: It follows from the equation (3.13) and (3.14) that

$$1 = \frac{1}{2} \left[\sum_{m=1}^{d_A} L_{\alpha\alpha}^{mm} + \sum_{m=1}^{d_A} L_{\beta\beta}^{mm} + \sqrt{-1} \sum_{m=1}^{d_A} L_{\alpha\beta}^{mm} - \sqrt{-1} \sum_{m=1}^{d_A} L_{\beta\alpha}^{mm} \right];$$

i.e.,

$$(3.21) \quad \sum_{m=1}^{d_A} L_{\alpha\beta}^{mm} - \sum_{m=1}^{d_A} L_{\beta\alpha}^{mm} = 0.$$

From the equations (3.20) and (3.21), we get $\sum_{m=1}^{d_A} L_{\alpha\beta}^{mm} = 0 (\alpha \neq \beta)$. In summary, $\sum_{m=1}^{d_A} L_{\mu\nu}^{mm} = \delta_{\mu\nu}$. \square

Note that the property (i) of the proposition 3.1. is not the condition of Hermiticity, and in general the matrix L representing the super-operator Φ is not Hermitian. However, by the definition of matrix realignment we can define the **dynamical matrix or Choi matrix** (see [13, 15]):

$$D_\Phi \equiv \mathcal{R}(L) \text{ with } D_{m\mu}^{n\nu} = L_{\mu\nu}^{mn}.$$

In particular, the mapping $\mathcal{J} : \Phi \mapsto D_\Phi$ is called *Choi-Jamiołkowski isomorphism*.

Proposition 3.2. *For a quantum channel Φ , its dynamical matrix D_Φ enjoy the properties that follow:*

- (i) $D_\Phi^\dagger = D_\Phi$;
- (ii) $D_\Phi \geq 0$;
- (iii) $\text{tr}_A(D_\Phi) = I_B$, $\text{Tr}(D_\Phi) = N$;
- (iv) $|L_\Phi\rangle\rangle = (I \otimes S \otimes I)|D_\Phi\rangle\rangle$; $\langle\langle L_\Phi|L_\Psi\rangle\rangle = \langle\langle D_\Phi|D_\Psi\rangle\rangle$; $\langle L_\Phi, L_\Psi \rangle = \langle D_\Phi, D_\Psi \rangle$;
- (v) $\langle\Phi(X), Y\rangle = \langle D_\Phi, Y \otimes X^* \rangle$ for any X, Y .

Proof. Write $D_\Phi = D = [D_{m\mu}^{n\nu}]$.

$$(i) \quad D^\dagger = [D_{m\mu}^{n\nu}]^\dagger = [D_{n\nu}^{m\mu}]^t = [D_{m\mu}^{n\nu}]^* = [L_{\mu\nu}^{mn}]^* = [L_{\mu\nu}^{mn}] = [D_{m\mu}^{n\nu}] = D.$$

(ii) Let $|z\rangle = \sum_{n,\nu=1}^N z_{n\nu} |n\nu\rangle$. Then $\langle z| = \sum_{m,\mu=1}^N z_{m\mu}^* \langle m\mu|$. Hence

$$\langle z|D|z\rangle = \sum_{m,\mu,n,\nu=1}^N z_{m\mu}^* D_{m\mu}^{n\nu} z_{n\nu}.$$

$|I\rangle\rangle = \sum_{m=1}^N |\mu\mu\rangle$ is called a maximally entangled state. So we have

$$|I\rangle\rangle\langle\langle I| = \sum_{\mu,\nu=1}^N |\mu\mu\rangle\langle\nu\nu| = \sum_{\mu,\nu=1}^N |\mu\rangle\langle\nu| \otimes |\mu\rangle\langle\nu|.$$

Since Φ is completely positive map, $\Phi \otimes \text{Id}_k \geq 0$ (\forall non-negative integer k), in particular, $(\Phi \otimes \text{Id}_N)(|I\rangle\rangle\langle\langle I|) \geq 0$, we get that

$$\begin{aligned} 0 &\leq \langle z|(\Phi \otimes \text{Id}_N)(|I\rangle\rangle\langle\langle I|)|z\rangle \\ &= \sum_{\mu,\nu=1}^N \langle z|[\Phi(|\mu\rangle\langle\nu|) \otimes |\mu\rangle\langle\nu|]|z\rangle \\ &= \sum_{\mu,\nu=1}^N \sum_{m,\alpha,n,\beta=1}^N z_{m\alpha}^* z_{n\beta} \langle m|\Phi(|\mu\rangle\langle\nu|)|n\rangle \cdot \langle\alpha|\mu\rangle\langle\nu|\beta\rangle \\ &= \sum_{m,\mu,n,\nu=1}^N z_{m\mu}^* z_{n\nu} L_{\mu\nu}^{mn} = \sum_{m,\mu,n,\nu=1}^N z_{m\mu}^* D_{m\mu}^{n\nu} z_{n\nu}. \square \end{aligned}$$

Obviously, there is an identity in the proof: if $\mathcal{D}(\mathcal{H}_B) \xrightarrow{\Phi} \mathcal{D}(\mathcal{H}_A)$, then $\mathcal{D}(\mathcal{H}_B \otimes \mathcal{H}_B) \xrightarrow{\Phi \otimes \text{Id}_N} \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$(3.22) \quad D_\Phi = (\Phi \otimes \text{Id}_N)(|I\rangle\rangle\langle\langle I|), \Phi(\rho) = \text{Tr}_A[D_\Phi(I_A \otimes \rho^t)].$$

Notes: If $X = |\mu\rangle\langle\nu|$, then $|X\rangle\rangle = |\mu\rangle|\nu\rangle \equiv |\mu\nu\rangle$, from which it follows that $|\Phi(X)\rangle\rangle = L|X\rangle\rangle = L|\mu\nu\rangle = \sum_{i,j=1}^N |ij\rangle\langle ij|L|\mu\nu\rangle = \sum_{i,j=1}^N L_{ij}^{\mu\nu} |ij\rangle$. Therefore, $\Phi(X) = \sum_{i,j=1}^N L_{ij}^{\mu\nu} |i\rangle\langle j|$, and

$$\begin{aligned}\langle m|\Phi(|\mu\rangle\langle\nu|)|n\rangle &= \langle m|\Phi(X)|n\rangle = \sum_{i,j=1}^N L_{ij}^{\mu\nu} \langle m|i\rangle\langle j|n\rangle \\ &= \sum_{i,j=1}^N L_{ij}^{\mu\nu} \delta_{mi}\delta_{nj} = L_{\mu\nu}^{mn}.\end{aligned}$$

Since $\rho_{\mu\nu} = \langle\mu|\rho|\nu\rangle = \text{tr}(\rho|\nu\rangle\langle\mu|) = \text{tr}(|\mu\rangle\langle\nu|\rho^\dagger)$, we have:

$$\begin{aligned}\Phi(\rho) &= \sum_{\mu,\nu} \rho_{\mu\nu} \Phi(|\mu\rangle\langle\nu|) = \sum_{\mu,\nu} \Phi(|\mu\rangle\langle\nu|) \text{tr}(|\mu\rangle\langle\nu|\rho^\dagger) = \sum_{\mu,\nu} \text{Tr}_A(\Phi(|\mu\rangle\langle\nu|) \otimes (|\mu\rangle\langle\nu|\rho^\dagger)) \\ &= \text{Tr}_A[(\Phi \otimes \text{Id}_N)(|I_B\rangle\langle I_B|)(I_A \otimes \rho^\dagger)] = \text{Tr}_A[D_\Phi(I_A \otimes \rho^\dagger)].\end{aligned}$$

(iii) Since $D = [D_{\mu\nu}]$, where $D_{\mu\nu} = [D_{mn}^{\mu\nu}]$, $\text{tr}_A D = [\text{tr} D_{\mu\nu}]$. Because $\text{tr} D_{\mu\nu} = \sum_{m=1}^N D_{mn}^{\mu\mu} = \sum_{m=1}^N L_{\mu\nu}^{mm} = \delta_{\mu\nu}$, thus we have $\text{tr}_A D = [\delta_{\mu\nu}] = I_B$. Furthermore, $\text{Tr}(D) = N$ is trivially.

(iv) By the operator-sum representation theorem, we have $\Phi(\rho) = \sum_j \Gamma_j \rho \Gamma_j^\dagger$, thus

$$\begin{aligned}|L_\Phi\rangle\rangle &= |\sum_j \Gamma_j \otimes \Gamma_j^*\rangle\rangle = \sum_j |\Gamma_j \otimes \Gamma_j^*\rangle\rangle = \sum_j (I \otimes S \otimes I) |\Gamma_j\rangle\rangle |\Gamma_j^*\rangle\rangle \\ &= \sum_j (I \otimes S \otimes I) |\Gamma_j\rangle\rangle \langle\langle \Gamma_j| \rangle\rangle = (I \otimes S \otimes I) |\sum_j \mathcal{R}(\Gamma_j \otimes \Gamma_j^*)\rangle\rangle \\ &= (I \otimes S \otimes I) |\mathcal{R}(\sum_j \Gamma_j \otimes \Gamma_j^*)\rangle\rangle = (I \otimes S \otimes I) |\mathcal{R}(L)\rangle\rangle \\ &= (I \otimes S \otimes I) |D_\Phi\rangle\rangle.\end{aligned}$$

Therefore $\langle\langle L_\Phi | L_\Psi \rangle\rangle = \langle\langle D_\Phi | (I \otimes S \otimes I)^2 | D_\Psi \rangle\rangle = \langle\langle D_\Phi | D_\Psi \rangle\rangle$; that is $\langle L_\Phi, L_\Psi \rangle = \langle D_\Phi, D_\Psi \rangle$.

(v)

$$\begin{aligned}\langle Y, \Phi(X) \rangle &= \langle\langle Y | \Phi(X) \rangle\rangle = \langle\langle Y | L_\Phi | X \rangle\rangle = \text{Tr}[L_\Phi | X] \langle\langle Y | \rangle\rangle \\ &= \text{Tr}[(|Y\rangle\rangle \langle\langle X|)^\dagger L_\Phi] = \langle\langle Y | \rangle\rangle \langle\langle X |, L_\Phi \rangle = \langle\mathcal{R}(|Y\rangle\rangle \langle\langle X|), \mathcal{R}(L_\Phi) \rangle \\ &= \langle Y \otimes X^*, D_\Phi \rangle.\end{aligned}$$

□

For any quantum channel Φ , it induces its *dual channel* Φ^\dagger in the following sense:

$$\langle\Phi(\rho), \sigma\rangle = \langle\rho, \Phi^\dagger(\sigma)\rangle \text{ for any states } \rho \text{ and } \sigma.$$

If a CP map is given by the Kraus form $\Phi(\rho) = \sum_j \Gamma_j \rho \Gamma_j^\dagger$, then the dual maps reads $\Phi^\dagger(\sigma) = \sum_j \Gamma_j^\dagger \sigma \Gamma_j$. Therefore, we have the following proposition into which the most useful results are summarized:

Proposition 3.3. (i) $L_\Phi = \sum_j \Gamma_j \otimes \Gamma_j^*$, or $D_\Phi = \sum_j |\Gamma_j\rangle\rangle \langle\langle \Gamma_j|$ for $\Phi(\cdot) = \sum_j \Gamma_j \cdot \Gamma_j^\dagger$.

(ii) If Φ^\dagger is the dual channel of a quantum channel Φ , then $L_{\Phi^\dagger} = \mathcal{F} \circ \mathcal{T}(L_\Phi) = L_\Phi^\dagger$, or $D_{\Phi^\dagger} = \mathcal{F} \circ \mathcal{T}(D_\Phi)$.

(iii) $L_{r\Phi+s\Psi} = rL_\Phi + sL_\Psi$, or $D_{r\Phi+s\Psi} = rD_\Phi + sD_\Psi$.

(iv) the composition $\Phi \circ \Psi$ of two maps Φ and Ψ means that $L_{\Phi \circ \Psi} = L_\Phi L_\Psi$, or $D_{\Phi \circ \Psi} = \mathcal{R}(\mathcal{R}(D_\Phi) \mathcal{R}(D_\Psi))$.

- (v) $L_{\mathcal{T} \circ \Phi} = \mathcal{F}_r(L_\Phi)$, or $D_{\mathcal{T} \circ \Phi} = \mathcal{T}_A(D_\Phi)$; $L_{\Phi \circ \mathcal{T}} = \mathcal{F}_c(L_\Phi)$, or $D_{\Phi \circ \mathcal{T}} = \mathcal{T}_B(D_\Phi)$.
 (vi) $L_{\mathcal{T} \circ \Phi \circ \mathcal{T}} = \mathcal{F}(L_\Phi) = L_\Phi^*$, or $D_{\mathcal{T} \circ \Phi \circ \mathcal{T}} = \mathcal{T}(D_\Phi) = D_\Phi^t = D_\Phi^*$.

Proof. (i) Since $L_\Phi|X\rangle\rangle = |\Phi(X)\rangle\rangle = |\sum_j \Gamma_j X \Gamma_j^\dagger\rangle\rangle = \sum_j \Gamma_j \otimes \Gamma_j^* |X\rangle\rangle$, $L_\Phi = \sum_j \Gamma_j \otimes \Gamma_j^*$. $D_\Phi = \mathcal{R}(L_\Phi) = \sum_j \mathcal{R}(\Gamma_j \otimes \Gamma_j^*) = \sum_j |\Gamma_j\rangle\rangle\langle\langle\Gamma_j|$.
 (ii) Obviously, $L_{\Phi^\dagger} = \sum_j \Gamma_j^\dagger \otimes \Gamma_j^t = (\sum_j \Gamma_j \otimes \Gamma_j^*)^\dagger = L_\Phi^\dagger$. Thus it follows from (3) of Proposition 2.9. that

$$\begin{aligned} D_{\Phi^\dagger} &= \mathcal{R}(L_{\Phi^\dagger}) = \mathcal{R}(L_\Phi^\dagger) = \mathcal{R} \circ \mathcal{T}(L_\Phi^*) \\ &= [\mathcal{R} \circ \mathcal{T}(L_\Phi)]^* = [\mathcal{F} \circ \mathcal{R}(L_\Phi)]^* = [\mathcal{F}(D_\Phi)]^* = \mathcal{F}(D_\Phi^*) = \mathcal{F} \circ \mathcal{T}(D_\Phi). \end{aligned}$$

(iii) It is trivially because

$$\begin{aligned} L_{r\Phi+s\Psi}|X\rangle\rangle &= |(r\Phi + s\Psi)(X)\rangle\rangle = r|\Phi(X)\rangle\rangle + s|\Psi(X)\rangle\rangle \\ &= rL_\Phi|X\rangle\rangle + sL_\Psi|X\rangle\rangle = (rL_\Phi + sL_\Psi)|X\rangle\rangle. \end{aligned}$$

$D_{r\Phi+s\Psi} = rD_\Phi + sD_\Psi$ holds since the reshuffle transformation is linear.

(iv) $L_{\Phi \circ \Psi}|X\rangle\rangle = |\Phi \circ \Psi(X)\rangle\rangle = L_\Phi|\Psi(X)\rangle\rangle = L_\Phi L_\Psi|X\rangle\rangle$. This implies that $D_{\Phi \circ \Psi} = \mathcal{R}(\mathcal{R}(D_\Phi)\mathcal{R}(D_\Psi))$.

(v) $L_{\mathcal{T} \circ \Phi} = L_{\mathcal{T}} L_\Phi = S L_\Phi = \mathcal{F}_r(L_\Phi)$; similarly, $L_{\Phi \circ \mathcal{T}} = L_\Phi L_{\mathcal{T}} = L_\Phi S = \mathcal{F}_c(L_\Phi)$. Thus

$$D_{\mathcal{T} \circ \Phi} = \mathcal{R}(L_{\mathcal{T} \circ \Phi}) = \mathcal{R} \circ \mathcal{F}_r(L_\Phi) = \mathcal{R} \circ \mathcal{F}_r \circ \mathcal{R}(D_\Phi) = \mathcal{T}_A(D_\Phi)$$

and

$$D_{\Phi \circ \mathcal{T}} = \mathcal{R}(L_{\Phi \circ \mathcal{T}}) = \mathcal{R} \circ \mathcal{F}_c(L_\Phi) = \mathcal{R} \circ \mathcal{F}_c \circ \mathcal{R}(D_\Phi) = \mathcal{T}_B(D_\Phi).$$

(vi) $L_{\mathcal{T} \circ \Phi \circ \mathcal{T}} = L_{\mathcal{T}} L_\Phi L_{\mathcal{T}} = S L_\Phi S = \mathcal{F}(L_\Phi)$. Thus

$$D_{\mathcal{T} \circ \Phi \circ \mathcal{T}} = \mathcal{R}(L_{\mathcal{T} \circ \Phi \circ \mathcal{T}}) = \mathcal{R} \circ \mathcal{F}(L_\Phi) = \mathcal{R} \circ \mathcal{F} \circ \mathcal{R}(D_\Phi) = \mathcal{T}(D_\Phi) = D_\Phi^t = D_\Phi^*. \quad \square$$

Proposition 3.4. For two quantum operations Φ, Ψ on the N -dimensional identical subsystems, $\mathcal{H}_A, \mathcal{H}_B$ of a bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$, respectively. Then:

$$(3.23) \quad L_{\Phi \otimes \Psi} = (I \otimes S \otimes I)(L_\Phi \otimes L_\Psi)(I \otimes S \otimes I).$$

Proof. $\rho = [\rho_{\mu\nu}] = \sum_{\mu, \nu=1}^N \rho_{\mu\nu} \otimes |\mu\rangle\langle\nu|$, where $\rho_{\mu\nu} = [\rho_{mn}^{\mu\nu}]$, is a $N \times N$ block density matrix whose entries being $N \times N$ scalar matrices. Since

$$\begin{aligned} (I \otimes S \otimes I)|\rho\rangle\rangle &= (I \otimes S \otimes I) \sum_{\mu, \nu=1}^N |\rho_{\mu\nu}\rangle\rangle \otimes |\mu\rangle\langle\nu| \\ &= \sum_{\mu, \nu=1}^N (I \otimes S \otimes I)^2 |\rho_{\mu\nu}\rangle\rangle \otimes |\mu\nu\rangle \\ &= \sum_{\mu, \nu=1}^N |\rho_{\mu\nu}\rangle\rangle \otimes |\mu\nu\rangle, \end{aligned}$$

we can get that

$$\begin{aligned}
L_{\Phi \otimes \Psi}|\rho\rangle\rangle &= |(\Phi \otimes \Psi)(\rho)\rangle\rangle = \sum_{\mu, \nu=1}^N |\Phi(\rho_{\mu\nu}) \otimes \Psi(|\mu\rangle\langle\nu|)\rangle\rangle \\
&= \sum_{\mu, \nu=1}^N (I \otimes S \otimes I)[|\Phi(\rho_{\mu\nu})\rangle\rangle \otimes |\Psi(|\mu\rangle\langle\nu|)\rangle\rangle] \\
&= \sum_{\mu, \nu=1}^N (I \otimes S \otimes I)[L_{\Phi}|\rho_{\mu\nu}\rangle\rangle \otimes L_{\Psi}|\mu\nu\rangle] \\
&= \sum_{\mu, \nu=1}^N (I \otimes S \otimes I)(L_{\Phi} \otimes L_{\Psi})[|\rho_{\mu\nu}\rangle\rangle \otimes |\mu\nu\rangle] \\
&= (I \otimes S \otimes I)(L_{\Phi} \otimes L_{\Psi})[\sum_{\mu, \nu=1}^N |\rho_{\mu\nu}\rangle\rangle \otimes |\mu\nu\rangle] \\
&= (I \otimes S \otimes I)(L_{\Phi} \otimes L_{\Psi})(I \otimes S \otimes I)|\rho\rangle\rangle.
\end{aligned}$$

□

Proposition 3.5. *Let Φ, Ψ be two quantum operations on \mathcal{H}_N . If ρ, σ are states in $\mathcal{H}_N \otimes \mathcal{H}_N$ and $\sigma = (\Phi \otimes \Psi)(\rho)$, then:*

$$(3.24) \quad \mathcal{R}(\sigma) = L_{\Phi}\mathcal{R}(\rho)L_{\Psi}^t.$$

Proof.

$$\begin{aligned}
|\mathcal{R}(\sigma)\rangle\rangle &= (I \otimes S \otimes I)|\sigma\rangle\rangle = (I \otimes S \otimes I)|(\Phi \otimes \Psi)(\rho)\rangle\rangle = (I \otimes S \otimes I)L_{\Phi \otimes \Psi}|\rho\rangle\rangle \\
&= (I \otimes S \otimes I)(I \otimes S \otimes I)(L_{\Phi} \otimes L_{\Psi})(I \otimes S \otimes I)|\rho\rangle\rangle = (L_{\Phi} \otimes L_{\Psi})|\mathcal{R}(\rho)\rangle\rangle \\
&= |L_{\Phi}\mathcal{R}(\rho)L_{\Psi}^t\rangle\rangle.
\end{aligned}$$

□

Lemma 3.6. *The composition of two completely positive linear super-operators $\Phi \circ \Psi$ is again completely positive.*

Proof. To see that $\Phi \circ \Psi$ is completely positive, it suffices to show $(\Phi \circ \Psi) \otimes \text{Id}_k$ is positive for any $k \in \mathbb{N}$. Obviously, $(\Phi \circ \Psi) \otimes \text{Id}_k = (\Phi \otimes \text{Id}_k) \circ (\Psi \otimes \text{Id}_k)$. Since Φ and Ψ are CP maps, $\Phi \otimes \text{Id}_k$ and $\Psi \otimes \text{Id}_k$ are positive for any $k \in \mathbb{N}$, which implies that the composition $(\Phi \otimes \text{Id}_k) \circ (\Psi \otimes \text{Id}_k)$ is positive for any $k \in \mathbb{N}$. □

Corollary 3.7. *Given two Hermitian matrices A, B of the same size $N^2 \times N^2$. If $A, B \geq 0$, then $\mathcal{R}(\mathcal{R}(A)\mathcal{R}(B)) \geq 0$.*

Proof. We can consider two non-negative matrices A and B of the same size $N^2 \times N^2$ as the dynamical matrices for two linear super-operators Φ_A and Φ_B , both acting from \mathbb{M}_N to \mathbb{M}_N , respectively. Now $A, B \geq 0$ imply that Φ_A and Φ_B are CP maps. Thus their composition $\Phi_A \circ \Phi_B$ is CP map by Lemma

3.6. It follows from this that, for $\Phi_A \circ \Phi_B$, its dynamical matrix $D_{\Phi_A \circ \Phi_B} = \mathcal{R}(\mathcal{R}(A)\mathcal{R}(B))$ is non-negative. The result that follows immediately. Another complicated proof on the present corollary can be found in [6]. \square

Corollary 3.8. *Given a finite set of Hermitian matrices $\{D_j : j = 1, \dots, n\}$ of the same size $N^2 \times N^2$. If $D_j \geq 0$ for all j , then $\mathcal{R}(\mathcal{R}(D_n)\mathcal{R}(D_{n-1}) \cdots \mathcal{R}(D_1)) \geq 0$.*

Proof. For each positive matrix D_j of size $N^2 \times N^2$, linear super-operator Φ_j determined by D_j is completely positive. Thus the composition of n completely positive linear super-operators $\{D_j : j = 1, \dots, n\}$ is denoted by $\Phi = \Phi_n \circ \cdots \circ \Phi_1$. Therefore the dynamical matrix D_Φ for Φ is equal to $\mathcal{R}(\mathcal{R}(D_n)\mathcal{R}(D_{n-1}) \cdots \mathcal{R}(D_1))$. Since composition preserves completely positivity by the above lemma, Φ is completely positive, therefore $D_\Phi \geq 0$. \square

Proposition 3.9. *The Hilbert-Schmidt inner product, i.e., $\langle X, Y \rangle = \text{Tr}(X^\dagger Y)$, on the matrix space \mathcal{M}_N induces another inner product in the space of linear maps $\mathcal{L}(\mathcal{M}_N, \mathcal{M}_N)$.*

Proof. Let $\{E_\alpha : \alpha = 1, \dots, N^2\}$ and $\{F_\alpha : \alpha = 1, \dots, N^2\}$ be orthonormal bases in \mathcal{M}_N , where $\langle E_\alpha, E_\beta \rangle = \langle F_\alpha, F_\beta \rangle = \delta_{\alpha\beta}$. We need only to prove that

$$\sum_{\alpha=1}^{N^2} \text{Tr} \Phi(E_\alpha)^\dagger \Psi(E_\alpha) = \sum_{\alpha=1}^{N^2} \text{Tr} \Phi(F_\alpha)^\dagger \Psi(F_\alpha).$$

Since $|E_\alpha\rangle\rangle = \sum_{\beta=1}^{N^2} |F_\beta\rangle\rangle \langle\langle F_\beta|E_\alpha\rangle\rangle = \sum_{\beta=1}^{N^2} \langle\langle F_\beta|E_\alpha\rangle\rangle |F_\beta\rangle\rangle$, $E_\alpha = \sum_{\beta=1}^{N^2} \langle\langle F_\beta|E_\alpha\rangle\rangle F_\beta$.

$$\begin{aligned} \sum_{\alpha=1}^{N^2} \text{Tr} \Phi(E_\alpha)^\dagger \Psi(E_\alpha) &= \sum_{\alpha=1}^{N^2} \sum_{\beta, \gamma=1}^{N^2} \overline{\langle\langle F_\beta|E_\alpha\rangle\rangle} \langle\langle F_\gamma|E_\alpha\rangle\rangle \text{Tr} \Phi(F_\beta)^\dagger \Psi(F_\gamma) \\ &= \sum_{\alpha=1}^{N^2} \sum_{\beta, \gamma=1}^{N^2} \langle\langle F_\gamma|E_\alpha\rangle\rangle \langle\langle E_\alpha|F_\beta\rangle\rangle \text{Tr} \Phi(F_\beta)^\dagger \Psi(F_\gamma) \\ &= \sum_{\beta, \gamma=1}^{N^2} \langle\langle F_\gamma|(\sum_{\alpha=1}^{N^2} |E_\alpha\rangle\rangle \langle\langle E_\alpha|) |F_\beta\rangle\rangle \text{Tr} \Phi(F_\beta)^\dagger \Psi(F_\gamma) \\ &= \sum_{\beta, \gamma=1}^{N^2} \langle\langle F_\gamma|F_\beta\rangle\rangle \text{Tr} \Phi(F_\beta)^\dagger \Psi(F_\gamma) = \sum_{\beta, \gamma=1}^{N^2} \delta_{\beta\gamma} \text{Tr} \Phi(F_\beta)^\dagger \Psi(F_\gamma) \\ &= \sum_{\alpha=1}^{N^2} \text{Tr} \Phi(F_\alpha)^\dagger \Psi(F_\alpha). \end{aligned}$$

\square

Now we define the inner product of two linear super-operators Φ and Ψ (see [2]) as follows:

$$(3.25) \quad \langle \Phi, \Psi \rangle \equiv \sum_{\alpha=1}^{N^2} \text{Tr} \Phi(E_\alpha)^\dagger \Psi(E_\alpha).$$

Using this correspondence it is possible to introduce two different bases, associated to the bases $\{E_\alpha\}_{\alpha=1}^{N^2}, \{F_\beta\}_{\beta=1}^{N^2}$, in the space of linear maps:

1) **Type-I basis** $\{\Delta_{\alpha\beta}\}$ in $\mathcal{L}(\mathcal{M}_N, \mathcal{M}_N)$ is defined by

$$(3.26) \quad \Delta_{\alpha\beta}(X) = E_\alpha \langle F_\beta, X \rangle = E_\alpha \text{Tr} F_\beta^\dagger X, \quad X \in \mathbb{M}_N;$$

and

2) **Type-II basis** $\{\Theta_{\alpha\beta}\}$ in $\mathcal{L}(\mathcal{M}_N, \mathcal{M}_N)$ is defined by

$$(3.27) \quad \Theta_{\alpha\beta}(X) = E_\alpha X F_\beta^\dagger, \quad X \in \mathbb{M}_N.$$

Indeed, 1) Let $\sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \Delta_{\alpha\beta} = 0$ for some scalars $c_{\alpha\beta} \in \mathbb{C}$. This implies that $\sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \Delta_{\alpha\beta}(X) = 0$, in particular, for $X = F_\gamma(\gamma = 1, \dots, N^2)$, we have:

$$0 = \sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \Delta_{\alpha\beta}(F_\gamma) = \sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \delta_{\beta\gamma} E_\alpha = \sum_{\alpha=1}^{N^2} c_{\alpha\gamma} E_\alpha$$

Since $\{E_\alpha\}$ is linearly independent, $c_{\alpha\gamma} = 0(\alpha, \gamma = 1, \dots, N^2)$. We have also that $\langle \Delta_{\alpha\beta}, \Delta_{\mu\nu} \rangle = \sum_{i,j,k,l=1}^{N^2} \text{Tr}[\Delta_{\alpha\beta}(|i\rangle\langle j|)^\dagger \Delta_{\mu\nu}(|k\rangle\langle l|)] = \delta_{\alpha\mu} \delta_{\beta\nu}$. Furthermore, $L_{\Delta_{\alpha\beta}} = |E_\alpha\rangle\langle\langle F_\beta|$.

2) Let $\sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \Theta_{\alpha\beta} = 0$ for some scalars $c_{\alpha\beta} \in \mathbb{C}$. This implies that $\sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \Theta_{\alpha\beta}(X) = 0$, we have:

$$0 = \sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \Theta_{\alpha\beta}(X) = \sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} E_\alpha X F_\beta^\dagger,$$

which means that

$$\begin{aligned} 0 &= \left| \sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} \Theta_{\alpha\beta}(X) \right\rangle = \sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} |\Theta_{\alpha\beta}(X)\rangle = \sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} |E_\alpha X F_\beta^\dagger\rangle \\ &= \left(\sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} E_\alpha \otimes F_\beta^* \right) |X\rangle; \end{aligned}$$

i.e., $\sum_{\alpha,\beta=1}^{N^2} c_{\alpha\beta} E_\alpha \otimes F_\beta^* = 0$ since X is arbitrary. Because of the independence of the set $\{E_\alpha \otimes F_\beta^*\}_{\alpha,\beta=1}^{N^2}$, this implies that $c_{\alpha\beta} = 0(\alpha, \beta = 1, \dots, N^2)$. And we have also that $\langle \Theta_{\alpha\beta}, \Theta_{\mu\nu} \rangle = \delta_{\alpha\mu} \delta_{\beta\nu}$. Furthermore, $L_{\Theta_{\alpha\beta}} = E_\alpha \otimes F_\beta^*$. \square

Remark 3.10. Therefore, according to two kind of the above-mentioned bases, we can expanding any mapping $\Phi \in \mathcal{L}(\mathcal{M}_N, \mathcal{M}_N)$ with respect to Type-I and Type-II bases, respectively, to get two expressions that follow:

$$(3.28) \quad \Phi = \sum_{\alpha,\beta=1}^{N^2} p_{\alpha\beta} \Delta_{\alpha\beta} = \sum_{\alpha,\beta=1}^{N^2} q_{\alpha\beta} \Theta_{\alpha\beta}.$$

Now $L_\Phi = \sum_{\alpha,\beta=1}^{N^2} p_{\alpha\beta} |E_\alpha\rangle\langle\langle F_\beta| = \sum_{\alpha,\beta=1}^{N^2} q_{\alpha\beta} E_\alpha \otimes F_\beta^*$. We write $P = [p_{\alpha\beta}]$, $Q = [q_{\alpha\beta}]$.

There is natural question to be asked: what is the relationships among these matrices P, Q ? (see [10])

Proposition 3.11. *With the above notations,*

$$(3.29) \quad \langle \Delta_{\alpha\beta}, \Theta_{\mu\nu} \rangle = \langle \Theta_{\alpha\beta}, \Delta_{\mu\nu} \rangle = \text{Tr}(E_\alpha^\dagger E_\mu F_\beta F_\nu^\dagger).$$

Thus

- (i) $p_{\alpha\beta} = \sum_{\mu,\nu=1}^{N^2} \text{Tr}(E_\alpha^\dagger E_\mu F_\beta F_\nu^\dagger) q_{\mu\nu};$
- (ii) $q_{\alpha\beta} = \sum_{\mu,\nu=1}^{N^2} \text{Tr}(E_\alpha^\dagger E_\mu F_\beta F_\nu^\dagger) p_{\mu\nu}.$

Proof. By the definition of the inner product in the space of linear maps,

$$\begin{aligned}
\langle \Delta_{\alpha\beta}, \Theta_{\mu\nu} \rangle &= \sum_{i,j=1}^N \text{Tr}((\Delta_{\alpha\beta}(|i\rangle\langle j|))^{\dagger} \Theta_{\mu\nu}(|i\rangle\langle j|)) = \sum_{i,j=1}^N \overline{\langle j|F_{\beta}^{\dagger}|i\rangle} \text{Tr}(E_{\alpha}^{\dagger} E_{\mu}|i\rangle\langle j|F_{\nu}^{\dagger}) \\
&= \sum_{i,j=1}^N \langle i|F_{\beta}|j\rangle \cdot \langle j|F_{\nu}^{\dagger} E_{\alpha}^{\dagger} E_{\mu}|i\rangle = \sum_{i=1}^N \langle i|F_{\beta} F_{\nu}^{\dagger} E_{\alpha}^{\dagger} E_{\mu}|i\rangle = \text{Tr}(F_{\beta} F_{\nu}^{\dagger} E_{\alpha}^{\dagger} E_{\mu}) \\
&= \text{Tr}(E_{\alpha}^{\dagger} E_{\mu} F_{\beta} F_{\nu}^{\dagger}).
\end{aligned}$$

Similarly, we have also: $\langle \Theta_{\alpha\beta}, \Delta_{\mu\nu} \rangle = \text{Tr}(E_{\alpha}^{\dagger} E_{\mu} F_{\beta} F_{\nu}^{\dagger})$. Since

$$\begin{aligned}
p_{\alpha\beta} &= \langle \Delta_{\alpha\beta}, \Phi \rangle = \sum_{\mu,\nu=1}^{N^2} \langle \Delta_{\alpha\beta}, \Theta_{\mu\nu} \rangle \langle \Theta_{\mu\nu}, \Phi \rangle \\
&= \sum_{\mu,\nu=1}^{N^2} \langle \Delta_{\alpha\beta}, \Theta_{\mu\nu} \rangle q_{\mu\nu},
\end{aligned}$$

1) and 2) is trivial. \square

Remark 3.12. (i) A special case is provided by the choice $E_{\alpha} = F_{\alpha}$ (or $E_{\alpha} = F_{\alpha} = |i\rangle\langle j|$, where $\{|i\rangle\}_{i=1}^N$ are an orthonormal basis for \mathbb{C}^N) (see [1]).

(ii) Since $|I\rangle\rangle = |\sum_i |i\rangle\langle i|| = \sum_i |ii\rangle$, $|I\rangle\rangle\langle\langle I| = \sum_{i,j} |ii\rangle\langle\langle jj| = \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|$. We know that $I \otimes I = \sum_{\alpha=1}^{N^2} |E_{\alpha}\rangle\rangle\langle\langle E_{\alpha}|$ when $\{E_{\alpha}\}_{\alpha=1}^{N^2}$ is an orthonormal basis for \mathcal{M}_N . Thus we have: $|I\rangle\rangle\langle\langle I| = \mathcal{R}(I \otimes I) = \sum_{\alpha=1}^{N^2} \mathcal{R}(|E_{\alpha}\rangle\rangle\langle\langle E_{\alpha}|) = \sum_{\alpha=1}^{N^2} E_{\alpha} \otimes E_{\alpha}^*$. If there is another orthonormal basis $\{F_{\beta}\}_{\beta=1}^{N^2}$ for \mathcal{M}_N , we still have: $|I\rangle\rangle\langle\langle I| = \sum_{\beta=1}^{N^2} F_{\beta} \otimes F_{\beta}^*$. Therefore, $\sum_{\alpha=1}^{N^2} E_{\alpha} \otimes E_{\alpha}^* = \sum_{\beta=1}^{N^2} F_{\beta} \otimes F_{\beta}^* = \sum_{i,j=1}^N |i\rangle\langle j| \otimes |i\rangle\langle j| = |I\rangle\rangle\langle\langle I|$. Furthermore, we have the swap operator $S = \sum_{i,j=1}^N |ij\rangle\langle\langle ji| = \sum_{\alpha=1}^{N^2} E_{\alpha} \otimes E_{\alpha}^{\dagger} = \sum_{\beta=1}^{N^2} F_{\beta} \otimes F_{\beta}^{\dagger}$.

(iii) In fact, given two orthonormal bases $\{E_{\alpha}\}_{\alpha=1}^{N^2}$ and $\{F_{\alpha}\}_{\alpha=1}^{N^2}$ in \mathcal{M}_N , the relation

$$(3.30) \quad \mathcal{L}(\mathcal{M}_N, \mathcal{M}_N) \ni \Phi \longrightarrow \Lambda_{\Phi} = \sum_{\alpha=1}^{N^2} \Phi(E_{\alpha}) \otimes F_{\alpha} \in \mathcal{M}_N \otimes \mathcal{M}_N$$

defines an isomorphism between $\mathcal{L}(\mathcal{M}_N, \mathcal{M}_N)$ and $\mathcal{M}_N \otimes \mathcal{M}_N$. The isomorphism is an isometry:

$$\begin{aligned}
\langle \Lambda_{\Phi}, \Lambda_{\Psi} \rangle &= \left\langle \sum_{\alpha=1}^{N^2} \Phi(E_{\alpha}) \otimes F_{\alpha}, \sum_{\beta=1}^{N^2} \Psi(E_{\beta}) \otimes F_{\beta} \right\rangle = \sum_{\alpha,\beta=1}^{N^2} \langle \Phi(E_{\alpha}) \otimes F_{\alpha}, \Psi(E_{\beta}) \otimes F_{\beta} \rangle \\
&= \sum_{\alpha,\beta=1}^{N^2} \langle \Phi(E_{\alpha}), \Psi(E_{\beta}) \rangle \langle F_{\alpha}, F_{\beta} \rangle = \sum_{\alpha,\beta=1}^{N^2} \langle \Phi(E_{\alpha}), \Psi(E_{\beta}) \rangle \delta_{\alpha\beta} \\
&= \sum_{\alpha=1}^{N^2} \langle \Phi(E_{\alpha}), \Psi(E_{\alpha}) \rangle = \langle \Phi, \Psi \rangle; \text{ i.e., } \langle \Lambda_{\Phi}, \Lambda_{\Psi} \rangle = \langle \Phi, \Psi \rangle.
\end{aligned}$$

4. BEST SEPARABLE APPROXIMATION FOR STATES

In this section we recall the so-called optimal and the best separability approximation (OSA and BSA respectively). Although the results below have been proven in [9, 7], we give the framework for our convenience. Other results involved can be found in [14]. In the method of BSA, for any density matrix ρ there exist a “optimal” separable matrix ρ_s^* and “optimal” non-negative scalar Λ such that $\rho - \Lambda\rho_s^* \geq 0$. We describe these results involved that follow:

Definition 4.1. A non-negative parameter Λ is called maximal with respect to a (not necessarily normalized) density matrix ρ , and the projection operator $P = |\psi\rangle\langle\psi|$ if $\rho - \Lambda P \geq 0$, and for every $\epsilon \geq 0$, the matrix $\rho - (\Lambda + \epsilon)P$ is not positive definite.

Definition 4.2. A pair of non-negative (Λ_1, Λ_2) is called maximal with respect to ρ and a pair of projection operators $P_1 = |\psi_1\rangle\langle\psi_1|$, $P_2 = |\psi_2\rangle\langle\psi_2|$, if $\rho - \Lambda_1 P_1 - \Lambda_2 P_2 \geq 0$, Λ_1 is maximal with respect to $\rho - \Lambda_2 P_2$ and to the projector P_1 , Λ_2 is maximal with respect to $\rho - \Lambda_1 P_1$ and to the projector P_2 and the sum $\Lambda_1 + \Lambda_2$ is maximal.

Theorem 4.3. For any density matrix ρ (separable, or not) and for any (fixed) countable set V of product vectors belonging to the range of ρ , there exist $\Lambda(V) \geq 0$ and a separable matrix

$$\rho_s^*(V) = \sum_{\alpha} \Lambda_{\alpha} P_{\alpha}$$

where each projector P_{α} is generated by some product vector in $R(\rho)$, and all $\Lambda_{\alpha} \geq 0$, such that $\delta\rho = \rho - \Lambda\rho_s^* \geq 0$, and that $\rho_s^*(V)$ provides the optimal separable approximation (OSA) to ρ since $\text{Tr}(\delta\rho)$ is minimal or, equivalently, Λ is maximal. There exists also the best separable approximation ρ_s^* for which $\Lambda = \max_V \Lambda(V)$. Obviously, $\Lambda(V) \leq \Lambda(V')$ when $V' \subset V$.

Theorem 4.4. Given the set V of product vectors in the range $R(\rho)$ of ρ , the matrix $\rho_s^* = \sum_{\alpha} \Lambda_{\alpha} P_{\alpha}$ is the optimal separable approximation (OSA) of ρ if:

- 1) all Λ_{α} are maximal with respect to $\rho_{\alpha} = \rho - \sum_{\alpha' \neq \alpha} \Lambda_{\alpha'} P_{\alpha'}$, and to the projector P_{α} ;
- 2) all pairs $(\Lambda_{\alpha}, \Lambda_{\beta})$ are maximal with respect to $\rho_{\alpha\beta} = \rho - \sum_{\alpha' \neq \alpha, \beta} \Lambda_{\alpha'} P_{\alpha'}$, and to the projection operators (P_{α}, P_{β}) .

Theorem 4.5. (The uniqueness of the BSA) Any density matrix ρ has a unique decomposition $\rho = \Lambda\rho_s + (1 - \Lambda)\delta\rho$, where ρ_s is a separable density matrix, $\delta\rho$ is a inseparable matrix with no product vectors in its range, and Λ is maximal.

5. BEST SEPARABLE APPROXIMATION FOR OPERATIONS

We can define separable CPM; that is, Φ is separable if its action can be expressed in the form

$$\Phi(\rho) = \sum_{i=1}^n (S_k \otimes T_k) \rho (S_i \otimes T_k)^{\dagger},$$

for some integer n and where S_k and T_k are operators acting on $\mathcal{H}_{A/B}$, respectively. Otherwise, we say that it is nonseparable. Up to proportionality constant, separable maps are those that can be implemented using local operations and classical communication only.

Let us consider two systems, A and B , spatially separated, each of them composed of two particles ($A_{1,2}$, and $B_{1,2}$). Let us consider a CPM Φ acting on systems A_1 and B_1 , where $\Phi(\rho) = \sum_k M^{(k)} \rho (M^{(k)})^\dagger$ and $M^{(k)}$ acting on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}$, where $M^{(k)} = S_k \otimes T_k$ for each index k . Now Φ induced another super-operator acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ in the following sense:

$$\widetilde{\Phi}(X) = \sum_k (\widetilde{S}_k \otimes \widetilde{T}_k) X (\widetilde{S}_k \otimes \widetilde{T}_k)^\dagger,$$

where $\widetilde{S}_k = S_k \otimes \text{id}^{(A_2)}$ and $\widetilde{T}_k = T_k \otimes \text{id}^{(B_2)}$, and X acting on $\mathcal{H}_A \otimes \mathcal{H}_B$.

We are interested in whether this CPM can create “nonlocal” entanglement between the systems A and B . We define the operator $E_{A_1 A_2, B_1 B_2}$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and $\mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$, and $\dim(\mathcal{H}_{A_i}) = \dim(\mathcal{H}_{B_i}) = d$, as follows:

$$E_{A_1 A_2, B_1 B_2} = (\Phi^{(A_1 B_1)} \otimes \text{id}^{(A_2 B_2)})(P_{A_1 A_2} \otimes P_{B_1 B_2}) \equiv \widetilde{\Phi}(P_{A_1 A_2} \otimes P_{B_1 B_2}) = \sum_k (\widetilde{S}_k \otimes \widetilde{T}_k)(P_{A_1 A_2} \otimes P_{B_1 B_2})(\widetilde{S}_k \otimes \widetilde{T}_k)^\dagger = \sum_k (\widetilde{S}_k P_{A_1 A_2} \widetilde{S}_k^\dagger) \otimes (\widetilde{T}_k P_{B_1 B_2} \widetilde{T}_k^\dagger).$$

Here, $P_{A_1 A_2} = |\Psi\rangle_{A_1 A_2} \langle \Psi|$ with $|\Psi\rangle_{A_1 A_2} = \frac{1}{\sqrt{d}} \sum_{m=1}^d |m\rangle_{A_1} \otimes |m\rangle_{A_2}$, and $P_{B_1 B_2} = |\Psi\rangle_{B_1 B_2} \langle \Psi|$ with $|\Psi\rangle_{B_1 B_2} = \frac{1}{\sqrt{d}} \sum_{\mu=1}^d |\mu\rangle_{B_1} \otimes |\mu\rangle_{B_2}$, where $\{|m\rangle : m = 1, \dots, d\}$ and $\{|\mu\rangle : \mu = 1, \dots, d\}$ are an orthonormal basis for \mathcal{H}_{A_1/A_2} and \mathcal{H}_{B_1/B_2} respectively. The map Φ is understood to act as the identity on the operators acting on \mathcal{H}_{A_2} and \mathcal{H}_{B_2} . The operator E has a clear interpretation since it is proportional to the density operator resulting from the operation Φ on systems A_1 and B_1 when both of them are prepared in a maximally entangled state with two ancillary systems, respectively. E is called *Choi matrix* for the bipartite super-operator Φ , or the mapping $\Phi \rightarrow E(\Phi)$ is called *Jamiołkowski isomorphism* for the bipartite super-operator Φ .

Now in general for $\Phi(\rho) = \sum_k M^{(k)} \rho (M^{(k)})^\dagger$, where $M^{(k)} = \sum_{mn, \mu\nu} M_{mn, \mu\nu}^{(k)} |m\rangle \langle n| \otimes |\mu\rangle \langle \nu|$, then Φ induced another super-operator like above as follows:

$$\widetilde{\Phi}(X) = \sum_k \widetilde{M}^{(k)} X (\widetilde{M}^{(k)})^\dagger,$$

where $\widetilde{M}^{(k)} = \sum_{mn, \mu\nu} M_{mn, \mu\nu}^{(k)} |\widetilde{m}\rangle \langle \widetilde{n}| \otimes |\widetilde{\mu}\rangle \langle \widetilde{\nu}|$; and $|\widetilde{m}\rangle \langle \widetilde{n}| = |m\rangle \langle n| \otimes \text{id}^{(A_2)}$ and $|\widetilde{\mu}\rangle \langle \widetilde{\nu}| = |\mu\rangle \langle \nu| \otimes \text{id}^{(B_2)}$.

$$\begin{aligned} E_{A_1 A_2, B_1 B_2} &= (\Phi^{(A_1 B_1)} \otimes \text{id}^{(A_2 B_2)})(P_{A_1 A_2} \otimes P_{B_1 B_2}) \\ &\equiv \widetilde{\Phi}(P_{A_1 A_2} \otimes P_{B_1 B_2}) = \sum_k \widetilde{M}^{(k)} (P_{A_1 A_2} \otimes P_{B_1 B_2}) (\widetilde{M}^{(k)})^\dagger \\ &= \sum_k \sum_{mm'nn', \mu\mu'\nu\nu'} M_{mn, \mu\nu}^{(k)} [M_{m'n', \mu'\nu'}^{(k)}]^* \\ &\quad (|m\rangle \langle n| \otimes \text{id}^{(A_2)}) P_{A_1 A_2} (|m'\rangle \langle n'| \otimes \text{id}^{(A_2)})^\dagger \otimes (|\mu\rangle \langle \nu| \otimes \text{id}^{(B_2)}) P_{B_1 B_2} (|\mu'\rangle \langle \nu'| \otimes \text{id}^{(B_2)})^\dagger \\ &= \sum_k \sum_{mm'nn', \mu\mu'\nu\nu'} M_{mn, \mu\nu}^{(k)} [M_{m'n', \mu'\nu'}^{(k)}]^* \\ &\quad (|m\rangle \langle n| \otimes \text{id}^{(A_2)} \otimes |\mu\rangle \langle \nu| \otimes \text{id}^{(B_2)})(P_{A_1 A_2} \otimes P_{B_1 B_2})(|m'\rangle \langle n'| \otimes \text{id}^{(A_2)} \otimes |\mu'\rangle \langle \nu'| \otimes \text{id}^{(B_2)})^\dagger. \end{aligned}$$

If we define $\text{vec}(M^{(k)}) = \sum_{mn, \mu\nu} M_{mn, \mu\nu}^{(k)} |m\rangle \langle n| \otimes |\mu\rangle \langle \nu| = \sum_{mn, \mu\nu} M_{mn, \mu\nu}^{(k)} |mn\rangle \otimes |\mu\nu\rangle = \sum_{mn, \mu\nu} M_{mn, \mu\nu}^{(k)} |mn\mu\nu\rangle$, then $\sum_k \text{vec}(M^{(k)}) \text{vec}(M^{(k)})^\dagger = E_{A_1 A_2, B_1 B_2}$. If $M^{(k)} = A_k \otimes B_k$, then $\text{vec}(M^{(k)}) = |A_k\rangle \rangle |B_k\rangle \rangle$.

Proposition 5.1. *If Φ is a quantum operation on a bipartite quantum system, then Φ is separable if and only if its dynamical matrix D_Φ is separable.*

Proof. By the definition of separable quantum operation, $\Phi(\rho) = \sum_i (A_i \otimes B_i) \rho (A_i \otimes B_i)^\dagger$ when Φ is separable. Now the dynamical matrix for the separable operation Φ is $D_\Phi = \sum_i \text{vec}(A_i \otimes B_i) \text{vec}(A_i \otimes B_i)^\dagger = \sum_i |A_i\rangle\langle A_i| \otimes |B_i\rangle\langle B_i|$. \square

Definition 5.2. Given quantum operation Φ on bipartite quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = N$, the dynamical matrix D_Φ for Φ can be decomposed as $D_\Phi = \lambda D_s + (1 - \lambda) D_e$ in terms of the BSA decomposition for state. Then the separable operation Φ_{BSA} determined by λD_s is called *best separable operation approximation* for Φ . $\Phi_{ENT} \equiv \Phi - \Phi_{BSA}$ is called *pure entanglement-produced operation part* for Φ .

Remark 5.3. If there is another decomposition $D_\Phi = D'_s + D'_e$ for which D'_s is just separable, then: $\lambda D_s - D'_s \geq 0$ by the uniqueness of the BSA. Thus the decomposition $\Phi = \Phi_{BSA} + \Phi_{ENT}$ is unique.

By operator-sum representation theorem, $\Phi(\rho) = \sum_{i \in \mathbb{F}} F_i \rho F_i^\dagger = \sum_{j \in \mathbb{G}} G_j \rho G_j^\dagger$, where $\max(|\mathbb{F}|, |\mathbb{G}|) \leq N^4$. Let

$$\mathbb{I} = \{i \in \mathbb{F} : F_i = A_i \otimes B_i\}, \mathbb{J} = \{j \in \mathbb{G} : G_j = C_j \otimes D_j\}.$$

Write $\Upsilon(\rho) = \sum_{i \in \mathbb{I}} F_i \rho F_i^\dagger$ and $\Psi(\rho) = \sum_{j \in \mathbb{J}} G_j \rho G_j^\dagger$; $\Upsilon'(\rho) = \sum_{i \in \mathbb{F} \setminus \mathbb{I}} F_i \rho F_i^\dagger$ and $\Psi'(\rho) = \sum_{j \in \mathbb{G} \setminus \mathbb{J}} G_j \rho G_j^\dagger$.

Theorem 5.4. $\Upsilon = \Psi = \Phi_{BSA}$.

Proof. Apparently, $D_\Phi = D_\Upsilon + D_{\Upsilon'}$, where D_Υ is separable since Υ is separable operation. Hence it follows from the uniqueness of the BSA that $\lambda D_s - D_\Upsilon \geq 0$. If, otherwise, $\lambda D_s - D_\Upsilon > 0$, then $D_{\Phi_{BSA} - \Upsilon} = D_{\Phi_{BSA}} - D_\Upsilon > 0$, that is, $\Phi_{BSA} - \Upsilon$ is CP map and separable, so $D_\Phi = [D_\Upsilon + D_{\Phi_{BSA} - \Upsilon}] + [D_{\Upsilon'} - D_{\Phi_{BSA} - \Upsilon}]$, where $D_{\Upsilon'} - D_{\Phi_{BSA} - \Upsilon} > 0$, contradict with the fact that there is no factorizing operational element for Υ' . Therefore $\lambda D_s - D_\Upsilon = 0$, equivalently, $\Upsilon = \Phi_{BSA}$. The theorem is proved. \square

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DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU, PEOPLE'S REPUBLIC OF CHINA

E-mail address: godyalin@163.com, linyz@zju.edu.cn